

Comment on “Localization Transition of Biased Random Walks on Random Networks”

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Sood and Grassberger studied in [1] random walks on random graphs that are biased towards a fixed target point. They put forward a critical bias strength b_c such that a random walker on an infinite graph eventually reaches the target with probability 1 when $b > b_c$, while a finite fraction of walks drift off to infinity for $b < b_c$. They rely on rigorous results obtained for biased walks on Galton-Watson (GW) trees [2] to calculate b_c , and give arguments indicating that this result should also hold for random graphs such as Erdos-Renyi (ER) graphs and Molloy-Reed (MR) graphs. To validate their prediction, they show by numerical simulations that the mean return time (MRT) on a finite ER graph, as a function of the graph size N , exhibits a transition around the expected b_c .

Here we show that the MRT on a GW tree can actually be computed analytically. This allows us (i) to show analytically that indeed the MRT displays a transition at b_c , (ii) to elucidate the N dependence of the MRT, which contradicts the $\propto N$ scaling expected in [1] for $b < b_c$.

Let us consider a realization of a GW tree of g generations, and denote by z_n the number of nodes of generation n ($0 \leq n \leq g-1$). The number of nodes at generation zero is taken equal to one ($z_0 = 1$) and each node has a random number of daughter nodes of mean k . As in ref. [1], we consider a random walker starting from the root node ($n = 0$) and experiencing a constant bias, such that the probabilities p_l^- and p_l^+ to jump from site l respectively towards and away from the root are given by $p_l^- = b/\mathcal{N}_l$ and $p_l^+ = b^{-1}/\mathcal{N}_l$ where \mathcal{N}_l is a normalization constant.

The key point of the derivation is that the MRT, here denoted by $\langle \mathbf{T} \rangle$, is given by the Kac formula [3, 4, 5]:

$$\langle \mathbf{T} \rangle = \frac{1}{P_{\text{eq}}(0)}, \quad (1)$$

where $P_{\text{eq}}(n)$ is the equilibrium distribution at genera-

tion n which is easily showed to verify $P_{\text{eq}}(n) \propto z_n b^{-2n}$. Normalization then yields straightforwardly

$$\langle \mathbf{T} \rangle = \sum_{n=0}^{g-1} z_n b^{-2n}. \quad (2)$$

We then denote by \overline{X} the average of a quantity X over the realizations of the graph. Using $\overline{z_n} = k^n$ [6], we first obtain the average number of nodes in the GW tree $N = (1 - k^g)/(1 - k)$, and finally the desired quantity

$$\overline{\langle \mathbf{T} \rangle} = \frac{1 - (1 + (k-1)N)^\epsilon}{1 - (k/b^2)} \quad (3)$$

with $\epsilon = \ln(k/b^2)/\ln(k)$.

Equation (3) clearly shows that $\overline{\langle \mathbf{T} \rangle}$ exhibits a transition when $\epsilon = 0$ or equivalently $b = b_c = \sqrt{k}$, in agreement with the numerical simulations of [1]. Furthermore, we obtain explicitly the asymptotics of $\overline{\langle \mathbf{T} \rangle}$ for large N :

$$\overline{\langle \mathbf{T} \rangle} \sim \begin{cases} \frac{(k-1)^\epsilon}{k/b^2 - 1} N^\epsilon & \text{for } \epsilon > 0 \ (b < b_c) \\ \frac{\ln N}{\ln k} & \text{for } \epsilon = 0 \ (b = b_c) \\ \frac{1 - (k-1)^\epsilon N^\epsilon}{1 - k/b^2} & \text{for } \epsilon < 0 \ (b > b_c) \end{cases} \quad (4)$$

Equations (3) and (4) show that in the case of an unbiased walk ($b = 1$), one has $\overline{\langle \mathbf{T} \rangle} = N$, but that this scaling does not hold as soon as $b \neq 1$. Note however that this exact scaling could be hard to distinguish from the numerical simulations of [1].

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